Burst process of stretched fiber bundles

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(Received 15 November 1993; revised manuscript received 17 March 1994)

A probabilistic delineation of the burst process of fiber bundles is proposed. It is shown that a burst process is governed by its rupture equation whose solution is fully characterized by the corresponding load function, which has a simple relation to the initial disorder. The extremes of the load function determine the criticalities of a burst process. According to burst size and influence on the whole bundle, the critical phenomena are divided into three categories: globally critical, subcritical, and quasicritical. As the number of fibers N in a bundle tends to infinity, the sizes of critical regions relative to N tend to zero. Rupture beyond critical regions is stable, whereas rupture in critical regions is rather unstable: A small increase of the external force may lead to an avalanche, i.e., a failure of a large number of fibers. Avalanches occur only in critical regions.

PACS number(s): 05.70.Jk, 02.50.−r, 62.20.Mk, 81.40.Np

I. INTRODUCTION

Rupture properties of random medium systems are interesting in both science and technology. Some models have been defined to describe various aspects of real situations [1,2]. Among them, fuse models have recently attracted much attention [3–10]. However, very few theoretical results have been obtained due to the complex interplay of disordered failures and the subsequent redistribution of local electrical currents (or, in mechanical term, local stresses). For example, in the study of fuse models, a large set of algebraic equations (Kirchhoff's law) must be solved to determine the new distribution of current in the fuse network after a fuse is broken, but one cannot predict *a priori* which fuse will fail. Computer simulations are commonly employed. However, in order to reveal the underlying "physics," some simplifications should be made. Although such simplified models may be less realistic, they can show us more about the universal and the intrinsic properties of the system, which are difficult to discover by using numerical methods alone.

As one of the simplest models of fracture, the failure of fiber bundles has been studied over many years since the original work in 1926 by Pierce [11]. The model assumes that all fibers in a bundle share equally the load borne on the bundle and that every fiber has a threshold of breakdown strength, which obeys some probability distribution. Stretched by an increasing force, the fibers in the bundle fail gradually and finally the bundle breaks. The model is equivalent to the parallel connected fuse model, in which the fuses have the same resistance but randomly distributed thresholds of current. Such models can be used to study the role of the initial disorder in the rupture without concerning complex spatial details.

Earlier work on the fiber-bundle model concerns only the strength of the entire bundle, which is usually associated with the theory of extreme statistics [12,13]. It was proved in 1945 by Daniels [14] that under some general conditions, the strength of a fiber bundle obeys asymptotically the central limit theorem. Additional statistical behaviors of the bundle strength under more relaxed restrictions or in more realistic situations were studied by several authors [15–21].

Now we may say that the bundle strength problem, which concerns only the "static" failure properties, has been thrashed out and reaches a rather clear understanding. In contrast, growth aspects of the rupture of fiber bundles have been studied only in recent years. Some scaling relations in the burst process of fiber bundles were discovered. Hemmer and Hansen [22] proved analytically that the burst-size distribution $D(\Delta)$ obeys a universal power law, namely, $D(\Delta)\propto\Delta^{-5/2}$, where Δ is the number of fibers that breaks simultaneously and $D(\Delta)$ is the probability of one failure event with burst size Δ. By computer simulations, Lu and Ding [23] showed that both the sizes and "locations" of large bursts exhibit scaling behaviors, where a "large" burst means the one with the largest size in all preceding bursts in the burst process. All fibers in the initial bundle are numbered from 1 to N consecutively in accordance with their strength thresholds from the smallest to the largest. The "location" of a burst is defined as the sequential number of the fiber that breaks first in this burst. It is worth noting that these scaling behaviors are, to a great extent, independent of the distribution of failure thresholds of individual fibers. Though being highly structure sensitive, rupture may generally be grasped in such simple ways. The discoveries of such universal properties of fracture encouraged us and provided a useful clue for further exploration.

Another universal property of fracture is criticality. Usually, as the stretching force increases gradually the fibers in the bundle fail bit by bit. When the force arrives at a certain value t_c, which is called the critical force or the mean strength of the whole bundle, the situation becomes quite different: A failure of any fiber in the n_c remaining fibers may trigger a large number of consequent failures. The final breakdown of the bundle is a sudden catastrophe with a great number of fibers failing together. The rupture before the critical point is stable; a

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small cause results in a small effect. While the rupture in the critical region is rather unstable, a microscopic event (a failure of few fibers) may lead to a macroscopic failure (a failure of a large number of fibers). The critical region is a region around t_c with a width of the order $O(\sqrt{N})$ [14,20]. The final failure with n_c fibers breaking together is usually the largest one. The ratios t_c/N and n_c/N tend to fixed values, which depend only on the strength distribution function of individual fibers in the thermodynamic limit $N \to \infty$. Of course, this is the most common situation and is also in agreement with actual experiences.

Are there any other situations of criticality? How does the failure of a fiber bundle evolve? What is the strength of a bundle in more general cases of the strength distribution of individual fibers? When does an avalanche occur? The motivation of this paper is to investigate these problems in detail. To study the properties of the whole burst process of a fiber bundle we shall begin with the following problem: When the bundle is stretched by a given force t, how many fibers will survive the break down? The model is described in Sec. II, where a general picture of the burst process is given. In Sec. III, the probability distribution of the number of the surviving fibers at a given external load is calculated by means of recurrence techniques. From the obtained recurrence relations, the asymptotic solution of the distribution in the thermodynamic limit $N \to \infty$ is derived in Sec. IV. The mean values of the number of the surviving fibers are calculated in Sec. V. Section VI is devoted to the investigation of the critical phenomena. Some numerical examples are given in Sec. VII. Finally, some remarks and discussions are included in Sec. VIII.

II. BURST ANALYSIS

Consider a load-carrying fiber bundle consisting of a total of N fibers. The fibers are clamped at both ends, one is fixed, and a stretching force t is applied on the other end (Fig. 1). The fibers in the bundle are assumed to have random strength thresholds X_j, $j=1,2,\ldots,N$, which are independent random variables with the same cumulative distribution function $F(x)$ and density function $f(x)$:

$$\mathrm{Prob}(X_j \le x) = F(x) = \int_0^x f(u)\,du \quad \forall j \tag{1}$$

("$\forall j$" reads as "for all j" and in this paper it means "for $j=1,2,\ldots,N$"). The total external load t on the bundle is supposed to be equally shared by all unbroken fibers in the bundle. Whenever a fiber bears a force equal to or greater than its strength threshold, the fiber breaks immediately and does not share any force then and after.

Suppose the external force increases slowly from 0 to infinity. When it reaches such a value t that the load shared by each fiber just equals the smallest strength threshold of the remaining fibers in the bundle, the weakest fiber fails and the force borne by each of the remaining fibers increases a little. Thus another fiber may fail at the same t. If so, the force applied to each of the other fibers will increase again and further failure may occur, and so on. This lasts until all remaining fibers no longer break at the same t. We call this event the "burst" that occurs at external force t. The burst size is defined as the number of fibers which fail simultaneously in this event.

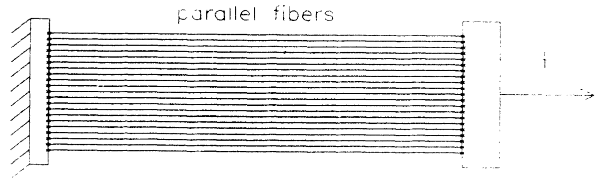

FIG. 1. A fiber bundle pulled by force t.

Rearrange the thresholds $\{X_j\}$ into ascending order as $\{X_j^*\}$ such that $X_1^* \le X_2^* \le \cdots \le X_N^*$. If $X_1^* \le t/N$ then the first fiber breaks and each of the remaining fibers bears tension $t/(N-1)$ consequently. At the same t, if one also has $X_2^* \le t/(N-1)$, then the second fiber breaks too. Generally, if the inequality $X_k^* \le t/(N-k+1)$ holds for $k=1,2,\ldots,m$, then the m weakest fibers in the bundle will break together. Moreover, if $X_{m+1}^* > t/(N-m)$ also holds at the same external force t, then the other fibers in the bundle will no longer break. In this case, there are exactly m fibers broken down.

Let

$$g_k = (N-k+1)X_k^* \quad \forall k \ . \tag{2}$$

The condition that there are exactly m fibers that have failed under external load t on the bundle is that g_{m+1} is the first term exceeding t in the series $g_1, g_2, \ldots, g_N$. A burst can only occur at the point that the external load equals some g_k. It is evident that a burst does not occur at every such point. If g_k is less than some preceding g_j ($j<k$), then the kth fiber must fail together with some of its preceding fibers. Hence the strength of the entire bundle is $\max g_k$. Figure 2 is a sample for a bundle with $N=100$ fibers whose strength thresholds obey uniform distribution $F(x)=x$ ($0 \le x \le 1$). In this sample we see that, for example, the 14th, 15th, and 16th fiber will fail together with the 13th fiber at the external load $t=g_{13}=14$. Because $g_{17} > g_{13}$, the 17th fiber will not fail together with fibers 13–16. The size of this burst is 4.

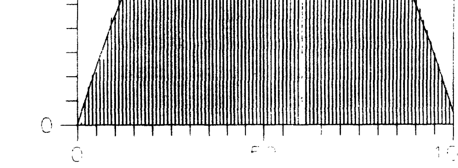

FIG. 2. $\{g_j\}$ of a sample bundle with $N=100$ fibers. The strength threshold of each fiber is uniformly distributed on the interval [0,1]. The curve is the load function multiplied by N: $NG(k)=k(1-k/N)$.

The strength of this bundle is $\max g_k = g_{46} = 27$.

According to the Kolmogorov-Smirnov theorem in mathematical statistics theory (see, for example, Ref. [24])

$$F(x_k) = \frac{k-1}{N} + O(1/\sqrt{N}) , \quad (3)$$

we have

$$g_k = N[G(X_k^*) + O(1/\sqrt{N})] \quad (4)$$

in which

$$G(x) = x[1-F(x)] . \quad (5)$$

On average, if each surviving fiber in the bundle shares a stretching force x, then the number of the survivals is asymptotically equal to $N[1-F(x)]$ and therefore the load on the bundle must be $Nx[1-F(x)] = NG(x)$. For this reason we call $G(x)$ the load function (per fiber). The curve in Fig. 2 is the corresponding load function (multiplied by N). The load function plays a crucial role in the burst process because of relation (4).

When we take the thermodynamic limit $N \to \infty$ the gaps between neighboring X_k^*'s tend to 0; thus we can treat the discrete values X_k^*'s in a continuous manner. In fact, it has already been proved that the bundle strength is asymptotically the maximum of the load function $G(x)$ provided that $G(x)$ has a unique maximum [14,20]. The load function dominates the average behavior in the rupture process.

Now we turn to investigate the influence of the fluctuations. Consider an interval (x', x'') in which $G(x)$ increases monotonically. On average, the terms of $\{g_j\}$ falling in the interval (Nx', Nx'') constitute an increasing series because of the relation (4). Therefore, when the external load falls in this interval, the bursts will be of size 1 only. In other words, no fiber will fail together with other fibers whose corresponding g_j's belong to the interval. This is obviously an incorrect failure picture. The reason is that an improper average has been done. In fact, the average of bursts should be performed after the statistics within each individual sample.

Suppose that g_k and $g_{k+\Delta}$ belong to the interval (Nx', Nx''), $\Delta > 0$. From Eq. (4), the fluctuations of g_K and $g_{k+\Delta}$ around $NG(X_k^*)$ and $NG(X_{k+\Delta}^*)$, respectively, are of the order $O(\sqrt{N})$. In addition to $G(X_{k+\Delta}^*) > G(X_k^*)$ if g_K can exceed $g_{k+\Delta}$ by a fluctuation, then the difference ΔG between $G(X_k^*)$ and $G(X_{k+\Delta}^*)$ should be of the order $O(1/\sqrt{N})$ at most. As N tends to infinity, if $G'(X_k^*)$ is not zero, the order of ΔG with respect to N is the same as $F(X_{k+\Delta}^*) - F(X_k^*)$ because of the relations

$$F(X_{k+\Delta}^*) - F(X_k^*) \simeq f(X_k^*)(X_{k+\Delta}^* - X_k^*) , \quad (6)$$

$$G(X_{k+\Delta}^*) - G(X_k^*) \simeq G'(X_k^*)(X_{k+\Delta}^* - X_k^*) . \quad (7)$$

Additionally, according to Eq. (3),

$$F(X_{k+\Delta}^*) - F(X_k^*) = \frac{\Delta}{N} + O(1/\sqrt{N}) . \quad (8)$$

Hence ΔG is of order $O(1/\sqrt{N})$ provided Δ is not greater than $O(\sqrt{N})$. So we have shown that in usual regions, the burst sizes do not exceed $O(\sqrt{N})$. It should be noted that if $G'(X_k^*) = 0$, the burst size may be larger than $O(\sqrt{N})$, but it can never reach the order $O(N)$ in this case.

Now we briefly discuss the distribution of the number of survivals under external load t. Let x be the force applied on each remaining fiber in the bundle. On average, it follows from the above argument that

$$t = NG(x) \quad (9)$$

and the survival number

$$n = N[1-F(x)] . \quad (10)$$

As t is given, the corresponding x can be obtained from Eq. (9) and the survival number is calculated using Eq. (10). It can be shown that the order $O(\sqrt{N})$ fluctuation of $\{g_j\}$ around $NG(x)$ leads to a same order fluctuation of the survival number around its mean, given by Eqs. (9) and (10).

In some cases Eq. (9) has several solutions. Which should be selected? In fact, in an actual sample bundle the survival number should be determined by the conditions

$$g_1, g_2, \ldots, g_{N-n} < t \; ; \; g_{N-n+1} > t . \quad (11)$$

Once the condition (11) is fulfilled, the other following terms of $\{g_j\}$ need not be considered. This means that only the largest candidate of n is correct, or only the smallest solution of Eq. (9) should be selected.

III. THE PROBABILITY DISTRIBUTION OF THE SURVIVAL NUMBER

For an ascending series $x_1 \le x_2 \le \cdots \le x_N$, one has

$$\text{Prob}(x_j \le X_j^* \le x_j + dx_j \quad \forall j)$$
$$= N! dF(x_1) dF(x_2) \cdots dF(x_N) . \quad (12)$$

The factor $N!$ appears in Eq. (12) because there are in total $N!$ disordered series corresponding to one ordered series $\{X_j^*\}$.

Set $p_n(t)$ denoting the probability that there are n fibers not broken down in the bundle stretched by total force t. For simplicity, let

$$t_k = \frac{t}{N-k+1} \quad \forall k .$$

According to the argument in Sec. II,

$$p_n(t) = \text{Prob}(X_1^* \le t_1, X_2^* \le t_2, \ldots, X_{N-n}^* \le t_{N-n},$$
$$X_{N-n+1}^* > t_{N-n+1}) . \quad (13)$$

From Eqs. (12) and (13) one can prove that (see Appendix A)

$$\sum_{j=m}^{N} \begin{bmatrix} j \\ m \end{bmatrix} \left[-F\left[\frac{t}{j}\right] \right]^{j-m} \left[1-F\left[\frac{t}{j}\right] \right]^{-j} p_j(t)=\delta_{mN} , \tag{14}$$

where $\begin{bmatrix} j \\ m \end{bmatrix}$ is the binomial coefficient $\begin{bmatrix} j \\ m \end{bmatrix} = j!/(j-m)!m!$, $\delta_{ij}=0$ if $i \neq j$ and $\delta_{ij}=1$ otherwise:

$$\sum_{j=0}^{N} z^{N-j} \left[\frac{1-zF\left[\frac{t}{j}\right]}{1-F\left[\frac{t}{j}\right]} \right]^{j} p_j(t)=1 . \tag{15}$$

Equation (14) is a linear recurrence relation for $\{p_j(t)\}$. Due to the complex dependence of the coefficient of $p_j(t)$ on j in Eq. (14), the equation cannot be solved exactly. In principle, one can numerically compute $p_N(t)$, $p_{N-1}(t), \ldots, p_0(t)$, by using Eq. (14) step by step when t is given. Unfortunately, however, the high numerical instability of the recurrence relation (14) makes it almost impossible to be used for any practical purpose when N is large. For example, the single precision (about 7 significant decimal digits) just meets the need of the computation for $N=40$, while the computation for $N=80$ needs the double precision (about 16 significant decimal digits).

$p_n(t)$ can be rewritten as

$$p_n(t)= \begin{bmatrix} N \\ n \end{bmatrix} Q_{N-n}(t) \left[1-F\left[\frac{t}{n}\right] \right]^{n} . \tag{16}$$

$Q_{N-n}(t)$ has a physical implication: It is the failure probability of $N-n$ selected fibers out of total N fibers in the bundle stretched by force t. Because the $N-n$ fibers have already failed, each of them must have a strength threshold smaller than t/n. Thus we have

$$Q_{N-n}(t) \le \left[1-F\left[\frac{t}{n}\right] \right]^{n} . \tag{17}$$

Therefore the probability $p_n(t)$ can be given as

$$p_n(t)=a_n(t) \begin{bmatrix} N \\ n \end{bmatrix} \left[F\left[\frac{t}{n}\right] \right]^{N-n} \left[1-F\left[\frac{t}{n}\right] \right]^{n} \tag{18}$$

with $a_n(t) \le 1$ for every n. Such a form of $p_n(t)$ is quite similar to that of a binomial distribution, except for the factor $a_n(t)$. The main difference is that the "trial probability" $F(t/n)$ varies with n. The factor $a_n(t)$ guarantees the normalization of $\{p_n(t)\}$. In analogy to binomial distributions, the important parts of the distribution $\{p_n(t)\}$ is concentrated in some small intervals of values of n.

IV. THE ASYMPTOTIC ANALYSIS OF THE PROBABILITY DISTRIBUTION

By simple consideration, one can point out directly that for small, but not null, values of n, $p_n(t)$ is very small. When t is small, the load shared by each fiber is also very small. So it is almost impossible to burst many fibers. The probability that there are only a few fibers left is then close to 0. When t is large, if there are a few fibers left in the bundle, then these fibers should have very large thresholds of strength and the probability of such situations arising may also be neglected.

$p_0(t)$ describes the global failure probability, which should be considered separately. Therefore we have to treat $p_0(t)$ and $p_n(t)$ with relative large n. For $n>0$, by using the Stirling formula, Eq. (18) can be converted to the form

$$p_n(t)=\frac{\sqrt{N}a_n(t)}{\sqrt{2\pi n(N-n)}} \exp\left\{ n \ln \frac{N\left[1-F\left[\frac{t}{n}\right]\right]}{n} + (N-n)\ln \frac{NF\left[\frac{t}{n}\right]}{n} \right\} . \tag{19}$$

In order to research the asymptotic behavior of the probability distribution, we introduce rescaled continuous variables ξ and τ to replace n and t, respectively,

$$\xi=\frac{n}{N} , \quad \tau=\frac{t}{N} .$$

The thermodynamic limit is taken as follows: N, n, and t all tend to infinity, but ξ and τ remain finite values. Replacing the summation over n by the integral over ξ, i.e., making the substitution

$$\sum_{n>0} \to \int_1^N dn \to \int_0^1 N\, d\xi$$

and denoting

$$A(\xi;\tau)=\frac{Na_n(t)}{\sqrt{n(N-n)}} ,$$

$$\rho(\xi;\tau)=Np_n(t)=\left[\frac{N}{2\pi}\right]^{1/2} A(\xi;\tau)\exp[-N\phi(\xi;\tau)] , \tag{20}$$

where

$$\phi(\xi;\tau)=\xi \ln \frac{\xi}{1-F\left[\frac{\tau}{\xi}\right]} + (1-\xi)\ln \frac{1-\xi}{F\left[\frac{\tau}{\xi}\right]} , \tag{21}$$

a new continuous distribution $\rho(\xi;\tau)$ is then introduced to replace $p_n(t)$. But it should be noticed that $\rho(0;\tau)$ does not correspond to $p_0(t)$, the probability of breaking down the bundle, as discussed above. Strictly speaking, one has

$$\int_0^1 d\xi\,\rho(\xi;\tau)=\sum_{n>0} p_n(t)=1-p_0(t)\ . \tag{22}$$

Suppose $z<1$. Equation (18) corresponds to

$$\left[\frac{N}{2\pi}\right]^{1/2}\int_0^1 d\xi\, A(\xi;\tau)\exp[-Nh(\xi;\tau,z)]=1\ , \tag{23}$$

where

$$\begin{aligned} h(\xi;\tau,z)&=\phi(\xi;\tau)-(1-\xi)\ln z-\xi\ln\frac{1-zF\left[\dfrac{\tau}{\xi}\right]}{1-F\left[\dfrac{\tau}{\xi}\right]}\\ &=\xi\ln\frac{\xi}{1-zF\left[\dfrac{\tau}{\xi}\right]}+(1-\xi)\ln\frac{1-\xi}{zF\left[\dfrac{\tau}{\xi}\right]}\ . \end{aligned} \tag{24}$$

Using the evident inequality for $y>0$,

$$\ln y\geq 1-\frac{1}{y}\ , \tag{25}$$

one can easily show that $h(\xi;\tau,z)\geq 0$. Further, because the equality in Eq. (25) holds only for $y=1$, we see that $h(\xi;\tau,z)$ takes its minima if and only if

$$\xi=1-zF\left[\frac{\tau}{\xi}\right]\ . \tag{26}$$

Recalling that $F(x)$ is monotonously increasing and $z<1$, it can be found that there do exist solutions of Eq. (26) satisfying $1-z<\xi<1$ provided $F(\tau)>0$. The cases that $F(\tau)=0$ are not interesting since such small forces cannot burst any fiber at all.

In the thermodynamic limit $N\to\infty$, the integral, of the left-hand side of Eq. (23) as a whole is determined by the integrals over the neighborhoods of all minimal points of $h(\xi;\tau,z)$. The integral of Eq. (23) can be calculated by the Laplace method (see Appendix B). If Eq. (26) has only one solution, we obtain

$$A(\xi;\tau)=\frac{1-\dfrac{z\tau}{\xi^2}f\left[\dfrac{\tau}{\xi}\right]}{\xi^{1/2}(1-\xi)^{1/2}}\ , \tag{27}$$

where z is evaluated by Eq. (26). But, unfortunately, Eq. (26) has more than one solution in general cases. Here we cannot extract $A(\xi;\tau)$ yet. From numerical computations we discovered that Eq. (27) holds in all cases provided that ξ is assigned as the largest solution of Eq. (26). In other words, only the largest maximum of $h(\xi;\tau,z)$ contributes to the integral of Eq. (23). We shall use this conjecture in this paper, whose validity may be verified by comparing its deductions with experiments. Since Eq. (15) fully characterizes $\{p_n(t)\}$ and Eq. (23) is equivalent to Eq. (15), it is demonstrated that $A(\xi;\tau)$ can be extracted from Eq. (23). However, the problem remains of how to prove the validity of Eq. (27) mathematically in general cases.

V. THE MEAN VALUE AND THE DEVIATION OF THE SURVIVAL NUMBER

Denote by $N\mu(\tau)$ the mean number of the surviving fibers in the bundle under external load $t=N\tau$:

$$\begin{aligned} N\mu(\tau)&=\sum_{n>0} np_n(t)\\ &=\int_0^1 d\xi\,\rho(\xi;\tau)N\xi\\ &=\left[\frac{N}{2\pi}\right]^{1/2}\int_0^1 N\xi A(\xi;\tau)\exp[-N\phi(\xi;\tau)]d\xi\ . \end{aligned} \tag{28}$$

Using the inequality (25), it is easy to show that $\phi(\xi;\tau)$ reaches its minima at $\xi=\eta(\tau)$, where η satisfies

$$\eta(\tau)=1-F\left[\frac{\tau}{\eta(\tau)}\right]\ . \tag{29}$$

Here we encounter again the multimaximum problem similar to that in Sec. IV: Eq. (29) may have more than one solution or $\eta(\tau)$ may be a multivalue function. Since $\phi(\xi;\tau)=h(\xi;\tau,z=1)$, we use the similar method that only the largest solution contributes to the integral in Eq. (28) as the limiting case $(z\to 1)$ discussed in Sec. IV. Thus, using the Laplace method (Appendix B) we obtain

$$\begin{aligned} N\mu(\tau)&=\left[\frac{N}{2\pi}\right]^{1/2}N\eta^*A(\eta^*;\tau)\left[\frac{2\pi}{N\phi_{\xi\xi}(\eta^*;\tau)}\right]^{1/2}\\ &=N\eta^* \end{aligned} \tag{30}$$

where η^* is the largest solution of Eq. (29). Hence we call Eq. (29) the mean-value equation, which must be satisfied by $\mu(\tau)$.

Similarly, the deviation $ND(\tau)$ of the number of remaining fibers in the bundle under external load $t=N\tau$ is calculated as

$$\begin{aligned} ND(\tau)&=\sum_{n>0}[n-N\mu(\tau)]^2p_n(t)\\ &=\left[\frac{N}{2\pi}\right]^{1/2}\int_0^1 N^2(\xi-\eta^*)^2A(\xi;\tau)\\ &\qquad\times\exp[-N\phi(\xi;\tau)]d\xi\\ &=N\frac{\mu(1-\mu)}{\left[1-\dfrac{\tau}{\mu^2}f\left[\dfrac{\tau}{\mu}\right]\right]^2}\ . \end{aligned} \tag{31}$$

We can also consider the failure of fibers in a bundle stretched by a given force $N\tau$ as follows. Initially the bundle has $N\mu_0=N$ fibers; each fiber bears a tension τ. All fibers with strength threshold smaller than τ will burst first; the number of these fibers is $NF(\tau)$. Thus the number of remaining fibers is reduced to $N\mu_1=N-NF(\tau/\mu_0)$ and each remaining fiber bears a tension τ/μ_1. Therefore the number of remaining fibers is further reduced to $N\mu_2=N-NF(\tau/\mu_1)$. Each decrease of the number of remaining fibers increases slightly the force loaded by each survival and hence further failure occurs. Generally, if the survival number is $N\mu_i$ after the ith repetition of the above virtual process, then $N\mu_{i+1}=N-NF(\tau/\mu_i)$ is the survival number of the $(i+1)$th repetition. If such a procedure arrives finally at a finite limit value of $\mu>0$, then the bundle will stay at the state with $N\mu$ remaining fibers. Evidently μ must be stabilized at the largest solution of the mean-value equation (35).

Let

$$\mu(\tau)=\frac{\tau}{x} \ . \tag{32}$$

Equation (31) can be rewritten as

$$\tau=G(x) \ . \tag{33}$$

Given τ, $\mu(\tau)$ can be evaluated by Eq. (32) by extracting x from Eq. (33) in advance. This is very similar to the situation in Sec. II, where we have shown that if Eq. (33) has several solutions, then the smallest one should be chosen. That is to say, the largest solution of Eq. (31) is the required one for the mean value. This agrees with the conjecture in Sec. IV. In contrast, if the conjecture is incorrect, then we cannot come to a result in agreement with the conclusion in Sec. II.

VI. CRITICALITIES

In Sec. V, the calculations assume that there exists at least one solution for the mean-value equation (29). Unlike Eq. (23), Eq. (29) has no solution when τ is large enough. Because Eq. (29) is equivalent to Eqs. (32) and (33), it is clear that Eq. (33) and hence Eq. (29) have a solution only if $\tau\le\tau_c$; otherwise they have no solution, where

$$\tau_c=\max G(x) \ . \tag{34}$$

τ_c is the maximal load (per fiber) the bundle can bear. Does it really mean that if Eq. (29) has no solution, then the whole bundle has broken down? By calculating the integral using the similar technique in Secs. IV and V, it follows from Eq. (22) that

$$p_0(t)=\begin{cases}0 & \text{if Eq. (29) has a solution}\\ 1 & \text{otherwise} \ .\end{cases} \tag{35}$$

Therefore whether Eq. (29) has a solution or not is indeed a criterion of breaking down the bundle. τ_c is then called the globally critical external load (per fiber). At this point, there will be

$$n_c=N\frac{\tau_c}{x_c} \tag{36}$$

fibers failing together, where x_c is the point at which $G(x)$ arrives at its greatest maximum. We see that the final burst of the bundle is a burst of $O(N)$ size, which differs fundamentally from the bursts caused by fluctuations whose sizes is up to the order $O(\sqrt{N})$ (see Sec. II). For this reason, we call a burst with size up to the order $O(\sqrt{N})$ microscopic and a burst with size of the order $O(N)$ macroscopic. If $G(x)$ has only one maximum, which is the most common situation, the final burst is the only macroscopic burst.

Here we point out the other two possibilities, which, though being somewhat unusual, are very interesting. (i) For some threshold distribution functions, point exist at which the bundle undergoes macroscopic bursts, which are almost the same as the final burst, except that the bundle does not break down completely. We call these points subcritical. This is roughly the case that the initial bundle consists of several kinds of fibers and the strength differences between the different kinds of fibers are so large that fibers of large strength begin to fail only after all fibers of small strength fail completely. (ii) For appropriate threshold distributions, another kind of point exists; the burst sizes at these points are between macroscopic and macroscopic. We call these points quasicritical.

Differentiating Eq. (29) we have

$$\frac{d\mu}{d\tau}=-\frac{\frac{\tau}{\mu}f\left(\frac{\tau}{\mu}\right)}{1-\frac{\tau}{\mu^2}f\left(\frac{\tau}{\mu}\right)} \ . \tag{37}$$

Using Eqs. (32) and (33), the formula can be written as

$$\frac{d\mu}{d\tau}=-\frac{G(x)}{G'(x)}f(x) \ , \tag{38}$$

where $x=\tau/\mu$. It results from Eq. (38) that if $G'(x)\neq 0$, then $\mu(\tau)$ is differentiable for τ, that is to say, a small increase of the external load τ causes a small failure of a few fibers. The rupture is stable in regions where $G'(x)\neq 0$. We notice that in these regions the root mean square fluctuations $[ND(\tau)]^{1/2}$ are of the order $O(\sqrt{N})$, which is in agreement with the previous general considerations (Sec. II). When $G'(x)$ approaches zero, $d\mu/d\tau\to-\infty$, the rupture becomes unstable: A small increase of the external load may lead to a big avalanche with a great many of fibers bursting. The points where $G'(x)=0$ are the critical points of the burst process. According to Eq. (31), we also have $D(\tau)\to\infty$ at critical points; the fluctuations at critical points are much larger than those in stable regions.

To research the rupture behaviors at critical points, we investigate the solution of the mean-value equation (29) in some detail. It is convenient to solve this equation graphically. At first, it is easy to depict the $\eta(\tau)$ vs τ curve according to the equivalent parameter form of Eq. (29)

$$\eta=\tau/x \ , \quad \tau=G(x) \tag{39}$$

by setting x varying from 0 to ∞. Figures 3 and 4 give examples for three specially designed bundles (see Sec. VII for details). The curves in Fig. 3 are load functions. $\eta(\tau)$ is plotted as the dashed lines in Fig. 4. We see that those $\eta(\tau)$'s are multivalue functions. In accordance with Sec. V, for a given τ, the mean value $\mu(\tau)$ should be the largest among the various values of η. The curve of $\mu(\tau)$ vs τ can then be drawn immediately following this principle. The curves of the mean values in Fig. 4 are plotted

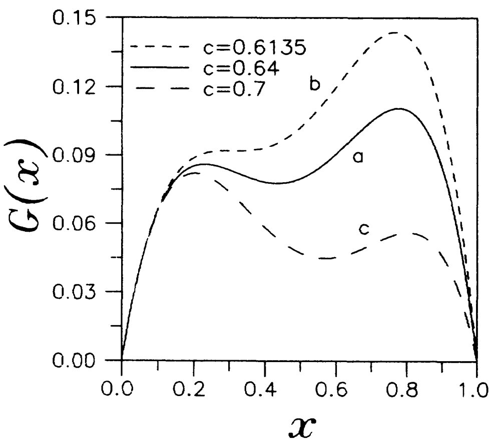

FIG. 3. The curves of the load function (42) of a special fiber bundle (Sec. VII) for three values of the parameter c. Curves a, b, and c correspond to $c=0.64$, $c=0.6135$, and $c=0.7$, respectively.

as solid lines.

Curves a, b, and c in Fig. 3 correspond to Figs. 4(a), 4(b), and 4(c), respectively. In Fig. 3, curve a has two maxima with the first one lower than the second; curve c is similar to curve a, except the first maximum is higher than the second; curve b has an inflection point followed by a maximum with the former lower than the latter. In Fig. 4(a) we see that $\mu(\tau)$ jumps down at $\tau=\tau_A$, which is the first maximum of curve a in Fig. 3. The bundle undergoes a macroscopic failure, but the whole bundle does not break down. This point is a subcritical point. The entire bundle breaks completely down at the globally critical point $\tau=\tau_B$. In Fig. 4(b), $\mu(\tau)$ has an infinitely small decreasing jump at $\tau=\tau_A$, which corresponds to the inflection point of curve b in Fig. 3. This is a quasicritical point at which $D(\tau)\rightarrow\infty$ [Eq. (29)] and the burst size is larger than the microscopic size, but smaller than the macroscopic size. The situation in Fig. 4(c) is rather simple since the whole bundle breaks down at point A, which corresponds to the first maximum of curve c in Fig. 3; the part of this curve following the greatest maximum does not influence the burst process. This is a general principle: The effective part of the load function governs the burst process, while the rest has no influence at all. The "effective part" of a load function $G(x)$ is defined as follows: A point x belongs to the effective part if and only if $G(x)\geq\sup_{u<x}G(u)$.

VII. NUMERICAL SIMULATIONS

To confirm the theoretical results obtained in the preceding sections, a special threshold distribution density function for individual fibers is designed. We consider a threshold distribution of the types of quadratic parabola defined in the interval [0,1]:

$$f(x)=\frac{3(x-c)^2}{(1-c)^3+c^3}\ , \quad x\in[0,1] \tag{40}$$

where $c<1$ is an adjustable parameter. This distribution has approximate bimodality: Its value is high at both

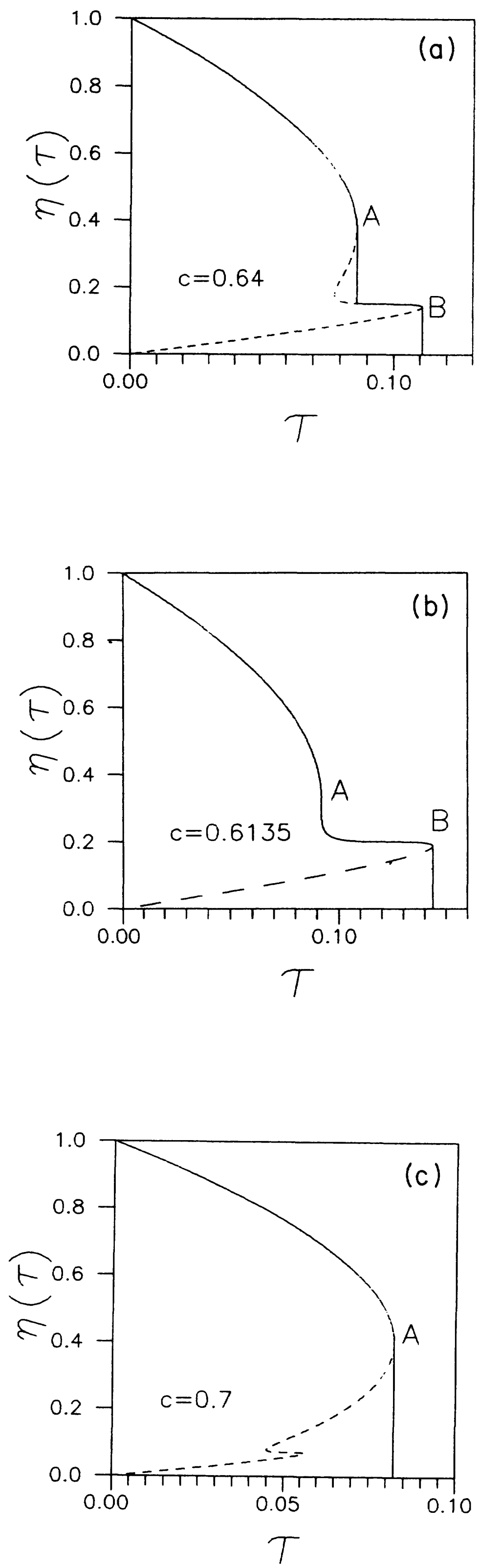

FIG. 4. The curves of $\eta(\tau)$ (dashed lines) and $\mu(\tau)$ (solid lines) of the designed bundle for the three selected values of c: (a) 0.64, (b) 0.6135, and (c) 0.7. The $\eta(\tau)$'s are multivalue functions. For a given τ, $\mu(\tau)$ is equal to the largest value of $\eta(\tau)$.

ends of the distribution interval and is low in the middle. The corresponding cumulative distribution function and load function are

$$F(x)=\frac{(x-c)^3+c^3}{(1-c)^3+c^3} \tag{41}$$

and

$$G(x)=\frac{x[(x-c)^3+c^3]}{(1-c)^3+c^3}\ , \tag{42}$$

respectively. Figure 3 is a plot of $G(x)$ with three selected values of c: (a) $c=0.64$, (b) $c=0.6135$, and (c) $c=0.7$. Using the theory of the cubic algebraic equation, it can be shown that $G(x)$ has an inflection point when $c=4^{1/3}/(1+4^{1/3})=0.6135\ldots$ The corresponding burst processes have already been analyzed in Sec. VI.

To simulate the burst process on a computer, pseudorandom numbers that obey the distribution (41) must be produced. It is easy to prove that the random variable Y obeys the distribution (41) if

$$Y=F^{-1}(X)=c+[(1-3c+3c^2)X-c^3]^{1/3}\ , \tag{43}$$

where X is a (0,1) uniformly distributed random variable. X can be obtained by a computer built-in procedure of random numbers and then Y will meet the requirement.

The simulations in this paper for each of the three cases are performed for 1000 samples. The initial number of fibers of each bundle is $N=1000$. The mean values and the deviations computed from the simulations are plotted in Figs. 5 and 6. For comparison, the theoretical curves are also depicted.

The burst sizes were counted during the simulations. The number of the bursts with size Δ is denoted by $W(\Delta)$. Figure 7 is the plot of $W(\Delta)$ vs Δ in the three cases. According to the result of Hemmer and Hansen [22],

$$W(\Delta)\propto\Delta^{-5/2} \tag{44}$$

holds for small bursts if the load function has a unique maximum. Our simulations show that the power law $W(\Delta)\propto\Delta^{-\gamma}$ holds for small bursts (up to size $\sqrt{N}$) too, in the cases that $G(x)$ has two maxima. The best fitting values of γ are around $\frac{5}{2}$. By our simulations, we expect that the power law (44) is universal if the number of fibers N is large enough.

VIII. CONCLUSIONS AND DISCUSSIONS

The evolution of the rupture of fiber bundles in the course of increasing external load is studied in this paper. Our investigations show that the load function plays a crucial role in the burst process of fiber bundles. First, the strength of a fiber bundle is determined by the greatest maximum of the load function. The result develops the previous work for fiber bundle strength [14,19,20], in which only single maximum load functions have been considered. We may expect that a similar central limit theorem holds for multimaxima cases, though it has not been deduced in this paper. Second, we show that the effective maxima of the load function determine

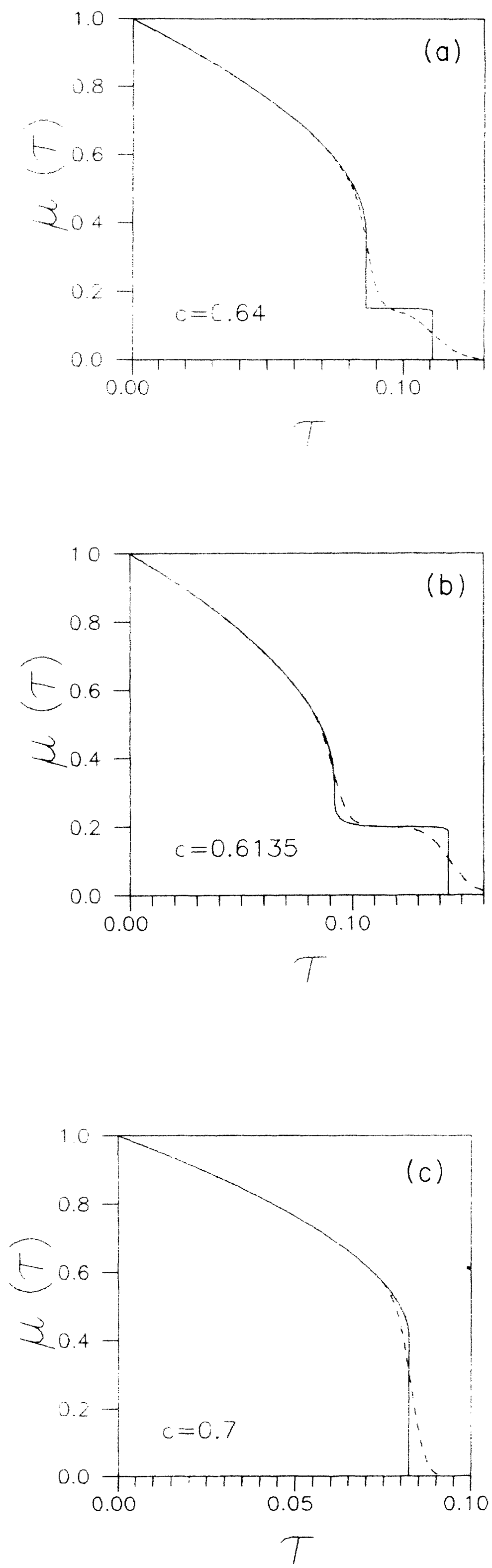

FIG. 5. The dashed curves are the mean values (divided by N) of the survival number. Each of them was computed from 1000 simulations for the designed models with each bundle containing $N=1000$ fibers. The solid lines are the corresponding theoretical curves. The values of c are the same as in Fig. 4.

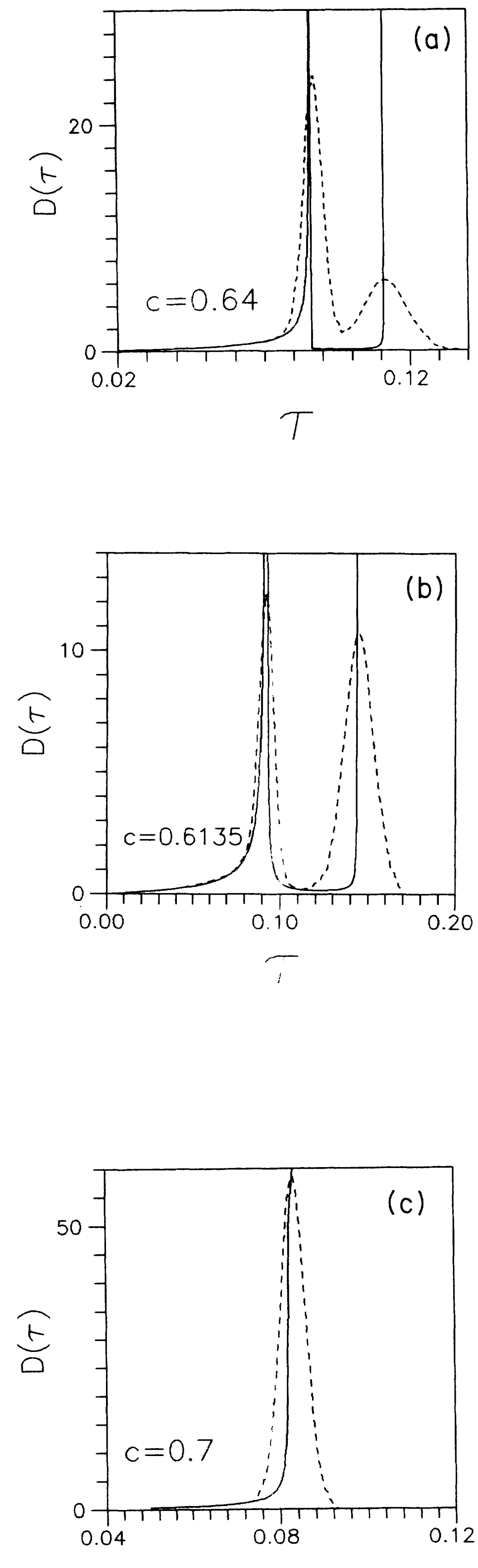

FIG. 6. The dashed curves are the deviations (divided by N) of the survival number computed from the same simulations as in Fig. 5. The solid lines are the corresponding theoretical curves, which have infinite peaks at the critical points.

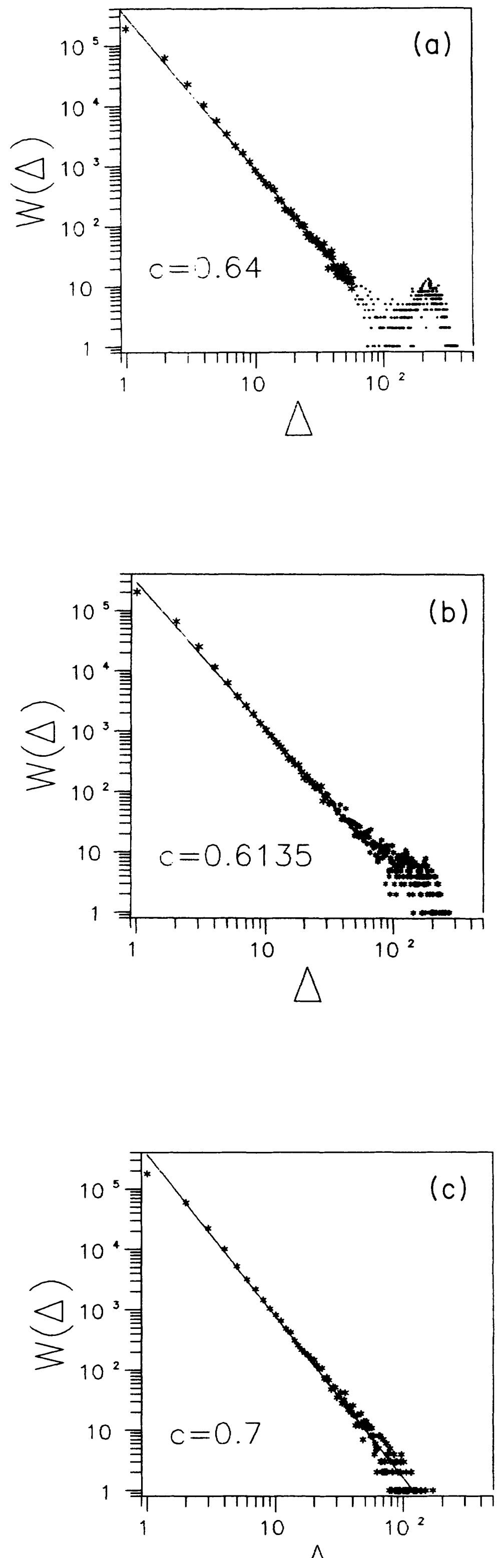

FIG. 7. The statistics of the burst sizes performed in the same simulations as in Figs. 5 and 6. The solid lines are the best fitting lines to the power law $W(\Delta) \propto \Delta^{-\gamma}$ for (a) $\Delta < 40$, (b) $\Delta < 60$, and (c) all Δ's. The best fitting values of γ are (a) $\gamma = 2.54$, (b) $\gamma = 2.43$, and (c) $\gamma = 2.69$.

the categories of criticalities, namely, global criticality, subcriticality, and quasicriticality.

It should be pointed out that for some fiber bundles there is no critical phenomenon provided the load functions have no extremes. The case $F(x)=\exp(-1/x)$ is an example. But it can be shown that if the threshold distribution for individual fibers has a second-order moment, then the load function must have an extreme. In fact, $1-F(x)$ must decrease more rapidly than $1/x$; hence $G(x)$ approaches 0 when $x \to \infty$. Therefore the criticality is a general feature of actual fiber bundles.

On the other hand, we have demonstrated that the $O(N)$ bursts occur only at critical points. All bursts that occur elsewhere are caused by fluctuations. The sizes of the latter are up to the order $O(\sqrt{N})$ and obey a power law [22]. The numerical simulations show that the power law holds for small-size bursts even in cases that the load functions have several maxima. It seems likely that power law is a universal principle for fluctuation phenomena.

ACKNOWLEDGMENT

The author is grateful to Professor Liaofu Luo for his going over of the manuscript and kind guidance.

APPENDIX A: PROOF OF EQ. (14)

$$\begin{aligned}
p_n(t) &= \mathrm{Prob}(X_1^* \le t_1, X_2^* \le t_2, \ldots, X_{N-n}^* \le t_{N-n}, X_{N-n+1}^* > t_{N-n+1}) \\
&= N! \left\{ \int_0^{t_1} dF(x_1) \int_{x_1}^{t_2} dF(x_2) \cdots \int_{x_{N-n-1}}^{t_{N-n}} dF(x_{N-n}) \right\} \\
&\quad \times \left\{ \int_{t_{N-n+1}}^{x_{N-n+2}} dF(x_{N-n+1}) \int_{t_{N-n+1}}^{x_{N-n+3}} dF(x_{N-n+2}) \cdots \int_{t_{N-n+1}}^{\infty} dF(x_N) \right\} \\
&= N! I_{N-n}(t) J_n(t) ,
\end{aligned} \tag{A1}$$

where $I_{N-n}(t)$ and $J_n(t)$ stand, respectively, for the $(N-n)$-fold integral in the first brace and the n-fold integral in the second. The calculation of $J_n(t)$ is given as (remembering that $t_{N-n+1}=t/n$)

$$\begin{aligned}
J_n(t) &= \int_{t/n}^{x_{N-n+3}} \left[F(x_{N-n+2}) - F\left[\frac{t}{n}\right] \right] dF(x_{N-n+2}) \cdots \int_{t/n}^{x_N} dF(x_{N-1}) \int_{t/n}^{\infty} dF(x_N) \\
&= \tfrac{1}{2} \int_{t/n}^{x_{N-n+4}} \left[F(x_{N-n+3}) - F\left[\frac{t}{n}\right] \right]^2 dF(x_{N-n+2}) \cdots \int_{t/n}^{x_N} dF(x_{N-1}) \int_{t/n}^{\infty} dF(x_N) \\
&= \cdots \\
&= \frac{1}{(n-1)!} \int_{t/n}^{\infty} \left[F(x_n) - F\left[\frac{t}{n}\right] \right]^{n-1} dF(x_n) \\
&= \frac{1}{n!} \left[1 - F\left[\frac{t}{n}\right] \right]^n .
\end{aligned} \tag{A2}$$

$I_k(t)$ cannot be expressed in a simple explicit form. We first deduce its recursive expression

$$\begin{aligned}
I_k &= \int_0^{t_1} dF(x_1) \int_{x_1}^{t_2} dF(x_2) \cdots \int_{x_{k-2}}^{t_{k-1}} dF(x_{k-1}) \int_{x_{k-1}}^{t_k} dF(x_k) \\
&= F(t_k) I_{k-1} - \int_0^{t_1} dF(x_1) \int_{x_1}^{t_2} dF(x_2) \cdots \int_{x_{k-2}}^{t_{k-1}} F(x_{k-1}) dF(x_{k-1}) \\
&= F(t_k) I_{k-1} - \tfrac{1}{2}[F(t_{k-1})]^2 I_{k-2} + \tfrac{1}{2} \int_0^{t_1} dF(x_1) \cdots \int_{x_{k-3}}^{t_{k-2}} [F(x_{k-2})]^2 dF(x_{k-2}) \\
&= F(t_k) I_{k-1} - \tfrac{1}{2}[F(t_{k-1})]^2 I_{k-2} + \cdots + \frac{(-1)^{k-2}}{(k-1)!}[F(t_2)]^{k-2} I_1 - \frac{(-1)^{k-2}}{(k-1)!} \int_0^{t_1} [F(x_1)]^{k-2} dF(x_1) \\
&= \sum_{j=1}^{k} \frac{(-1)^{j-1}}{j!} [F(t_{k-j+1})]^j I_{k-j} ,
\end{aligned} \tag{A3}$$

where it has been assumed that $I_0 \equiv 1$. Let $m = N-k$. The recurrence relation (A3) can be equivalently written as

$$\sum_{j=m}^{N} \frac{1}{(j-m)!}\left[-F\left[\frac{t}{j}\right]\right]^{j-m} I_{N-j}(t) = \delta_{mN} . \quad \text{(A4)}$$

Together with Eqs. (A1)–(A3), Eq. (14) is proven. Applying the "operator" $\sum_{m=0}^{N} z^{-m}$ to both sides of Eq. (14). Eq. (15) is then given, where z is an arbitrary number other than 0. Obviously, when $z=1$, Eq. (15) gives the normalization condition.

APPENDIX B: GENERALIZED LAPLACE ASYMPTOTIC INTEGRAL THEOREM

If $\psi(x)$ reaches its unique maximum $\psi(x_0)=0$ in the interval (a,b) at $x=x_0$, $g(x_0)\neq 0$, and $\psi''(x_0)<0$, then the following asymptotic relation holds for arbitrary $c > -1$ when $\lambda \to \infty$:

$$\int_a^b g(x)|x-x_0|^c \exp[\lambda\psi(x)]dx \sim \left[\frac{-2}{\lambda\psi''(x_0)}\right]^{(1+c)/2} \Gamma\left[\frac{1+c}{2}\right] g(x_0) . \quad \text{(B1)}$$

(See, for example, Ref. [25].)

According to this theorem, supposing that Eq. (25) has solutions ξ_i, $i=1,2,\ldots,m$, one gets

$$\mathcal{L} = \left[\frac{N}{2\pi}\right]^{1/2} \sum_{i=1}^{m} A(\xi_i;\tau)\left[\frac{2\pi}{N h_{\xi\xi}(\xi_i;\tau,z)}\right]^{1/2} , \quad \text{(B2)}$$

where $\mathcal{L}$ denotes the left-hand side of (23) and

$$h_{\xi\xi}(\xi;\tau,z) = \frac{\partial^2 h}{\partial\xi^2}(\xi;\tau,z) = \frac{\left[1-\frac{z\tau}{\xi^2} f\left[\frac{\tau}{\xi}\right]\right]^2}{\xi(1-\xi)} . \quad \text{(B3)}$$

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